Velocity distribution for a dilute vibrated granular material

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(Received 14 September 1998)

The velocity distribution for a vibrated granular material is determined in the dilute limit where the frequency of particle collisions with the vibrating surface is large compared to the frequency of binary collisions. The particle motion is driven by the source of energy due to particle collisions with the vibrating surface, and two dissipation mechanisms−inelastic collisions and air drag—are considered. In the latter case, a general form for the drag force is assumed. First, the distribution function for the vertical velocity for a single particle colliding with a vibrating surface is determined in the limit where the dissipation during a collision due to inelasticity or between successive collisions due to drag is small compared to the energy of a particle. In addition, two types of amplitude functions for the velocity of the surface, symmetric and asymmetric about zero velocity, are considered. In all cases, differential equations for the distribution of velocities at the vibrating surface are obtained using a flux balance condition in velocity space, and these are solved to determine the distribution function. It is found that the distribution function is a Gaussian distribution when the dissipation is due to inelastic collisions and the amplitude function is symmetric, and the mean square velocity scales as $[\langle U^2\rangle_S/(1-e^2)]$, where $\langle U^2\rangle_S$ is the mean square velocity of the vibrating surface and e is the coefficient of restitution. The distribution function is very different from a Gaussian when the dissipation is due to air drag and the amplitude function is symmetric, and the mean square velocity scales as $(\langle U^2\rangle_S g/\mu_m)^{1/(m+2)}$ when the acceleration due to the fluid drag is $-\mu_m u_y|u_y|^{m-1}$, where g is the acceleration due to gravity. For an asymmetric amplitude function, the distribution function at the vibrating surface is found to be sharply peaked around $[\pm 2\langle U\rangle_S/(1-e)]$ when the dissipation is due to inelastic collisions, and around $\pm[(m+2)\langle U\rangle_S g/\mu_m]^{1/(m+1)}$ when the dissipation is due to fluid drag, where $\langle U\rangle_S$ is the mean velocity of the surface. The distribution functions are compared with numerical simulations of a particle colliding with a vibrating surface, and excellent agreement is found with no adjustable parameters. The distribution function for a two-dimensional vibrated granular material that includes the first effect of binary collisions is determined for the system with dissipation due to inelastic collisions and the amplitude function for the velocity of the vibrating surface is symmetric in the limit $\delta_I \equiv (2nr)/(1-e) \ll 1$. Here, n is the number of particles per unit width and r is the particle radius. In this limit, an asymptotic analysis is used about the limit where there are no binary collisions. It is found that the distribution function has a power-law divergence proportional to $|u_x|^{(c\delta_I-1)}$ in the limit $u_x \to 0$, where u_x is the horizontal velocity. The constant c and the moments of the distribution function are evaluated from the conservation equation in velocity space. It is found that the mean square velocity in the horizontal direction scales as $O(\delta_I T)$, and the nontrivial third moments of the velocity distribution scale as $O(\delta_I \epsilon_I T^{3/2})$ where $\epsilon_I = (1-e)^{1/2}$. Here, $T = [2\langle U^2\rangle_S/(1-e)]$ is the mean square velocity of the particles. [S1063-651X(99)06703-3]

PACS number(s): 81.05.Rm, 45.05.+x, 05.70.Ln, 62.90.+k

I. INTRODUCTION

External vibrations have been widely used to assist the transport of granular materials in solids handling operations. In these operations, there is a transfer of energy from the vibrating surface to the particles, and this could result in the fluidization of the particles. In a fluidized state, the weight of the particles is balanced by the momentum transmitted due to instantaneous binary collisions between particles, and the energy transmitted from the vibrating surface to the particles is dissipated due to inelastic particle collisions or the fluid drag exerted by the surrounding gas. Vibrated granular materials have also been of interest because they exhibit unusual properties, such as gaslike, liquidlike, and solidlike states, and the propagation of density waves in the medium [1,2]. These types of behavior suggest that the macroscopic properties of these systems could be derived from the microscopic laws using techniques from statistical mechanics and the kinetic theory of gases. In the present paper, the average properties of a vibrated material are studied in the dilute ‘‘Knudsen’’ limit, where the frequency of binary collisions between particles is small compared to the frequency of particle collisions with the vibrating surface. The velocity distribution in the complementary limit, where the frequency of binary collisions is large compared to that of particle collisions with the vibrating surface, was the subject of earlier studies [9,10].

Gas fluidized beds, where the fluidization takes place due to the drag force caused by a flowing gas, have been of interest for some time. These have been traditionally described using continuum approaches, where the particle and gas phase are described using macroscopic mass and momentum equations. Constitutive relations, similar to those for compressible Newtonian fluids, are written for the two phases, and the coupling between the two phases is included in the form of a drag force that depends on the volume fraction and relative velocities of the two phases [3]. The stability of the uniformly fluidized state was first analyzed by

Jackson [4] using a simple constitutive relation, and it was found that the uniformly fluidized state is always unstable to density fluctuations. Since then, there have been many studies that have included more complicated constitutive relations for the stress tensor [5]. However, there has not yet been a consensus on the correct form of the macroscopic equations. Another approach to the derivation of macroscopic equations is to write equations for the particle motion, and use statistical averaging techniques to derive the macroscopic balance equations. This is complicated in a gas fluidized bed due to the complex nature of the interactions between the particles and the gas, and the turbulent flow of the gas. The description for a vibrofluidized bed is simpler due to the absence of the gas flow.

There has recently been a lot of interest in the velocity fluctuations of spatially uniform vibrated granular materials. There have been experimental and computer simulation studies which have tried to determine the scaling of the velocity fluctuations in the vibrated bed as a function of the frequency and the amplitude of oscillations of the vibrating surface. Luding, Herrmann, and Blumen [6] carried out ''event driven'' simulations of a two-dimensional system of inelastic disks in a gravitational field vibrated from below, and obtained scaling laws for the density variations in the bed. Their simulations indicate that the density of the particles decreases exponentially with height at large heights above the vibrating surface, and the height of the center of mass was found to vary as $H \propto U_0^{1.5}$, where U_0 is the amplitude of the velocity of the vibrating surface. Warr, Huntley, and Jacques [7] carried out an experimental study of a two-dimensional vibrofluidized bed. The density and velocity distribution functions were determined using image analysis techniques. Their experimental study also reported an exponential dependence of the density on the height near the top of the bed, similar to the Boltzmann distribution for the density of a gas in a gravitational field. However, the dependence of the density deviates from the exponential behavior near the bottom. The mean square velocity was found to vary as $T \propto U_0^{1.41}$ in the experimental study, and the height of the center of mass was found to vary as $H \propto U_0^{1.4}$. Luding [8] carried out simulations of rough two-dimensional disks, where the distribution of energy between the rotational and kinetic modes was examined as a function of the coefficients of restitution and friction. He observed that the power laws for the variation of the height of the center of mass with the number of particles and velocity of the vibrating surface from these simulations is different from that observed in the experiments [7]. Most theoretical studies predict a scaling $T \propto U_0^2$ [7], which is at variance with the experimental results.

The author used a kinetic theory analysis to study the velocity fluctuations in a vibrated granular material where the dissipation of energy is due to inelastic collisions or due to the fluid drag of the gas [9,10]. The limit where the frequency of binary collisions between particles is large compared to that of collisions with the vibrating surface was considered. In addition, the dissipation of energy during a collision due to inelasticity or between successive collisions due to drag is small compared to the energy of a particle, so that it is possible to use a perturbation analysis in which the particles are considered elastic in the leading approximation. In this case, the system resembles a hard sphere gas of elastic particles, the leading order density distribution is an exponentially decaying Boltzmann distribution, and the leading order velocity distribution is a Maxwell-Boltzmann distribution. However, the temperature of the system cannot be obtained from the leading approximation, but must be determined by a balance between the source of energy due to particle collisions with the vibrating surface and the dissipation of energy due to inelastic collisions or drag. The analysis indicated that the temperature scales as $T \propto U_0^2$ when the dissipation is due to inelastic collisions and $T \propto U_0^{4/3}$ when the dissipation is due to a drag force that is linear in the particle velocity. A perturbation to the leading order distribution function, in the form of a finite sum of the lowest nontrivial velocity moments, was used to calculate the effect of dissipation. The anisotropy and the skewness in the distribution function due to dissipation were found to be in qualitative agreement with the experiments [7]. The temperature scaling $T \propto U_0^{1.41}$ could be caused by the combined effect of inelastic collisions and drag. However, this analysis does not explain the scaling law $T \propto U_0^{1.5}$ observed in the simulations. Moreover, the exponential decay of the velocity profile is in error near the bottom of the bed, where the density is sufficiently large that an ideal gas law for the relation between pressure and temperature is not applicable.

The present analysis considers the complementary limit where the frequency of binary collisions between particles is small compared to the frequency of particle collisions with the vibrating surface. The system is very dilute, and is not likely to correspond to real applications. But the motivation for this analysis is as follows. One would expect a realistic model for a vibrofluidized bed to be applicable over a range of parameter values, ranging from dilute to dense. However, with the current analytical techniques available, one cannot obtain exact solutions to the Boltzmann equation that is applicable over a wide range of parameter values, and there is a compromise between the exactness of the analysis and its range of applicability. One could construct a phenomenological model for the behavior of a vibrofluidized bed, but for a consistent model it should agree with some exact solutions in asymptotic limits where such solutions can be obtained. One obvious limit is a dense system with low inelasticity where the distribution function is close to a Maxwell-Boltzmann distribution. Even though this limit is not encountered in technological situations, it is still valuable because it provides a reference point for less exact solutions that are valid over a larger parameter regime. In the same spirit, the present analysis is an attempt to obtain an exact solution in the opposite limit, so that models can be constructed that span the intermediate regime and are consistent with the exact limiting solutions. Similar calculations for dilute granular materials have been carried out previously by Kumaran and Koch [11] and Kumaran [12].

In the leading approximation, the binary collisions are neglected, and the distribution function is derived for a single particle colliding on a vibrating surface in the limit where the dissipation of energy during a collision is small compared to the energy of the particle. A single particle colliding with a vibrating surface was analyzed by Warr and Huntley [13] and Warr *et al.* [14], who used both experiments and computer simulations to determine the distribution function. The simulations indicated that the distribution function is a Gaussian distribution for a system where the dissipation of energy is due to inelastic collisions with the vibrating surface, and they used a generalized Langevin equation for the particle velocity to calculate the distribution function.

In Sec. II, the distribution function for a single particle colliding with a vibrating surface in a gravitational field is studied. Two dissipation mechanisms—inelastic collisions and fluid drag—are considered. In addition, two types of amplitude functions for the velocity of the vibrating surface—a symmetric amplitude function with zero mean velocity and a nonzero mean square velocity, and an asymmetric amplitude function with nonzero mean velocity—are considered. A differential equation for the distribution function is derived in the limit where the dissipation of energy during a collision due to inelasticity or between successive collisions due to drag is small compared to the energy of a particle. This equation is solved to obtain the distribution function. It is found that the form of the distribution function is sensitive to the type of dissipation and the type of amplitude function of the vibrating surface. The results of the analysis are verified using computer simulations, and excellent agreement is found with no adjustable parameters. For the case of inelastic collisions, a perturbation expansion in the parameter $\epsilon_I=(1-e)^{1/2}$ is used to determine the distribution function. Though earlier studies [7,15] have also used the parameter $(1-e^2)$ in their expansions, the two are essentially the same because $(1-e^2)=2(1-e)$ in the leading approximation for $(1-e)\ll 1$.

In Sec. III, the effect of binary collisions on the distribution function for a system with inelastic collisions and a symmetric amplitude function is analyzed using methods similar to those used in Kumaran and Koch [11] and Kumaran [12]. An asymptotic analysis is used in the limit $\delta_I \equiv (2nr)/(1-e)\ll 1$, where the particle distribution function is close to the distribution function in the absence of binary collisions. It is found that the mean square velocity in the horizontal direction scales as $O(\delta_I T)$, and the nontrivial third moments of the velocity distribution scale as $O(\delta_I(1-e)T^{3/2})$, where T is the mean square velocity in the vertical direction.

II. SINGLE-PARTICLE DISTRIBUTION FUNCTION

A. Dissipation due to inelastic collisions

The distribution function for the velocity of a particle in a gravitational field driven by a vibrating surface is derived in the present section. A two-dimensional coordinate system is used for the analysis, where the y axis is directed opposite to gravity, the x axis is in the horizontal plane, and the position of the vibrating surface varies symmetrically about the $y=0$ plane. The velocity of the vibrating surface is periodic, but no assumption is made regarding the exact form of the amplitude function. The collisions between the particle and the vibrating surface are inelastic with a coefficient of restitution e. The collision of the particle with the vibrating surface results in the transfer of energy from the surface to the particle, while the inelastic nature of the collision results in the dissipation of energy. It can be shown that when the coefficient of elasticity is close to 1 $[(1-e)\ll 1]$, the mean square velocity of the particle is large compared to the mean square velocity of the vibrating surface. In addition, it is assumed that the time period of oscillation of the surface is small compared to the time period between successive collisions, so that there is no correlation in the velocity of the surface at successive collisions. If the magnitude of the velocity of the vibrating surface is U, the increase in the kinetic energy of the particle due to a collision with the vibrating surface is $O(U^2)$, while the dissipation of energy due to inelasticity is $O((1-e^2)u_y^2)$, where u_y is the magnitude of the velocity of the particle. Equating the source and dissipation of energy, it is seen that the magnitude of the particle velocity is $u_y=O(U/(1-e^2)^{1/2})\gg U$, and an asymptotic analysis in the small parameter $\epsilon_I=(1-e)^{1/2}$ is used to determine the distribution function. In addition, it is assumed that the amplitude of oscillations of the vibrating surface is small compared to the maximum height of the particle $(u_y^2/2g)$, and the frequency of oscillations of the surface is large compared to the time between successive collisions of the particle, so that the interaction between the particle and the vibrating surface is modeled as a series of collisions with the plane located with $y=0$ with velocity U, and there is no correlation between the velocity of the plane during successive collisions.

In this section, a differential equation is derived for the distribution function $F(u_y)$, which is defined such that $F(u_y)du_y$ is the probability that the velocity of a particle that is leaving the vibrating surface is in the interval du_y about u_y. The distribution function $F(u_y)$ is defined only for $u_y>0$, and the distribution function for the velocity at any height can be inferred from $F(u_y)$, since the particle executes ballistic motion between successive collisions. Consider a collision between a particle with an initial velocity $-u_y'$ with the vibrating plate which has an instantaneous velocity U'. The velocity of the particle after the collision, u_y, is related to the initial velocity $-u_y'$ by

$$u_y-U'=-e(-u_y'-U'). \tag{1}$$

Note the requirement $U>-u_y'$ for a particle to collide with the surface.

Particle collisions with the vibrating surface cause a change in the particle velocity and a flux in velocity space. There is a collisional accumulation in the velocity interval du_y about u_y due to collision of the particle with velocity $-u_y'$ with the vibrating surface, where u_y' and u_y are related by Eq. (1). There is a collisional depletion in the interval du_y about u_y due to the collision of the particle with the velocity $-u_y$ with the surface. A differential equation for $F(u_y)$ at steady state is determined from the condition that *the ''average'' accumulation rate in the interval du_y about u_y, due to collisions of the particle with velocity in the interval du_y' about $-u_y'$ with the vibrating surface, is equal to the ''average'' depletion rate in the interval du_y about u_y due to collisions of the particle with velocity in the interval du_y about $-u_y$ with the vibrating surface*. Here, the term ''average'' denotes an average over the distribution of velocities of the vibrating surface, and u_y' and u_y are related by Eq. (1).

The rate of collision $\mathcal{R}$ of a particle with velocity in the interval du_y' about $-u_y'$ with the vibrating surface moving with velocity U' is

$$\mathcal{R}=(U'+u_y')F(u_y')du_y'. \tag{2}$$

The ''average'' flux of particles entering the interval du_y about u_y is then given by

$$N_{\mathrm{in}}(u_y)du_y=\langle(U'+u_y')F(u_y')du_y'\rangle_S, \tag{3}$$

where $\langle\,\rangle_S$ is an average over the distribution of velocities of the vibrating surface U'. Using Eq. (1) to express u_y' in terms of u_y, the flux of particles entering the interval du_y is

$$N_{\mathrm{in}}(u_y)=(1/e)\langle(U'+u_y')F(u_y')\rangle_S. \tag{4}$$

The rate of collision of the particle with velocity in the interval du_y about $-u_y$ with the vibrating surface is

$$\mathcal{R}=(u_y+U)F(u_y)du_y, \tag{5}$$

where U is the instantaneous velocity of the surface. The ''average'' flux of particles out of the interval du_y about u_y is

$$N_{\mathrm{out}}(u_y)du_y=\langle(u_y+U)F(u_y)du_y\rangle_S. \tag{6}$$

At steady state, the distribution function is determined from the equation $N_{\mathrm{in}}(u_y)=N_{\mathrm{out}}(u_y)$. This equation is difficult to solve in general, but a solution can be obtained in the limit $\epsilon_I \ll 1$. It was shown a little earlier that in this limit, the velocity of the mean square velocity of the surface $\langle U^2\rangle_S$ is small compared to the mean square velocity of the particle $\langle u_y^2\rangle$ and the difference in velocity due to a particle collision with the surface is small compared to the particle velocity. Consequently, it is necessary to expand the expression for the flux (3) in a series in the parameter $(u_y'-u_y)$,

$$F(u_y')=F(u_y)+(u_y'-u_y)\frac{dF}{du_y}+\frac{(u_y'-u_y)^2}{2}\frac{d^2F}{du_y^2}+O(u_y'-u_y)^3. \tag{7}$$

The above expansion is inserted into the flux balance condition $N_{\mathrm{in}}(u_y)=N_{\mathrm{out}}(u_y)$, and the resulting equation is expanded in the parameter ϵ_I and the velocity of the surface U'. The result, correct to $O(\epsilon_I)$ and $O(\langle U^2\rangle_S)$, is

$$\epsilon_I\left(2Fu_y+u_y^2\frac{dF}{du_y}\right)-2\langle U\rangle_S\left(u_y\frac{dF}{du_y}+2F\right)+2\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2}+\frac{dF}{du_y}\right)=0. \tag{8}$$

The solution for the conservation equation (8) depends on the amplitude function for the velocity of the vibrating surface. For a *symmetric* amplitude function, the average $\langle U\rangle_S=0$, and the conservation equation reduces to

$$\epsilon_I\left(2Fu_y+u_y^2\frac{dF}{du_y}\right)+2\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2}+\frac{dF}{du_y}\right)=0. \tag{9}$$

It can be easily verified that the normalized solution of the above equation for the distribution function F is a Gaussian distribution

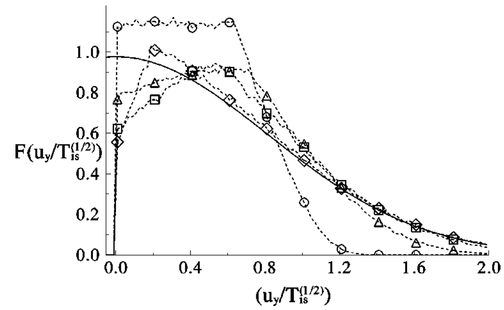

FIG. 1. Velocity distribution $F(u_y/\sqrt{T_{is}})$ as a function of $u_y/\sqrt{T_{is}}$ for a single particle colliding with a vibrating surface with a symmetric amplitude function, where dissipation is due to inelasticity. Here, T_{is} is given by Eq. (11). Solid line, analytical result (10); ○, $\epsilon_I=0.7$; △, $\epsilon_I=0.3$; □, $\epsilon_I=10^{-1}$; ◇, $\epsilon_I=10^{-2}$.

$$F=\sqrt{\frac{2}{\pi T_{is}}}\exp\left(-\frac{u_y^2}{2T_{is}}\right), \tag{10}$$

where the mean square velocity T_{is} is

$$T_{is}=(2\langle U^2\rangle_S/\epsilon_I). \tag{11}$$

The above expression indicates that the mean square velocity of the particles is proportional to $\langle U^2\rangle_S/\epsilon_I$, in agreement with the assumption made above regarding the relative magnitudes of the mean square velocities of the particles and the vibrating surface. In addition, the velocity distribution function is predicted to be a one-dimensional Maxwell-Boltzmann distribution at the vibrating surface.

The predicted distribution function (10) and (11) was verified using computer simulation studies of the particle collisions with a vibrating surface. In the simulation, the particle was given an initial velocity, and the velocity of the particle was updated at each collision using the collision rule (1). The velocity U of the vibrating surface was generated using a random number generator at each collision. For this purpose, a random number x was generated in the interval $0\leq x\leq 1$, and the velocity of the surface was $U=(2x-1)$ so that $\langle U\rangle_S=0$ for a symmetric amplitude function. In addition, care was taken to ensure that the constraint $U>-u_y$ [discussed after Eq. (1)] is satisfied. The velocity was first updated for 10^5 collisions without sampling to remove dependence on the initial particle velocity at the beginning of the simulation, and samples were then taken for another 10^5 collisions in order to determine the distribution function. The velocity coordinate in the range $-2\sqrt{T_{is}}\leq u_y\leq 2\sqrt{T_{is}}$ was divided into 200 bins of equal velocity intervals, and the distribution function at the center of the respective interval is proportional to the number of particles in each bin. The results of the simulation for $\epsilon_I=10^{-1},10^{-2}$, and 10^{-3} are compared with the analytical distribution function (10) and (11) in Fig. 1. From this figure it is seen that there is excellent agreement between the asymptotic result and the simulations for $\epsilon_I=10^{-2}$ and 10^{-1}. There is a deviation of about 10% of the maximum value of $F(u_y)$ for $\epsilon_I=0.3$. The deviation becomes about 25% of the maximum value for $\epsilon_I=0.7$, but the qualitative form of the distribution function is correctly captured by the asymptotic analysis. In addition, there is a difference between the predicted and measured distribution function near $u_y=0$ because the asymptotic analysis, which assumes $u_y \gg U$, becomes inaccurate for $u_y \sim U$.

The distribution function at any height y is obtained by solving the steady-state Boltzmann equation in the absence of binary collisions between particles:

$$\frac{\partial(u_y f)}{\partial y}+\frac{\partial(a_y f)}{\partial u_y}=0, \qquad (12)$$

where the acceleration $a_y=-g$, $f(y,u_y)$ is the distribution function which is defined so that $f(y,u_y)dy\,du_y$ is the number of particles per unit width with vertical position in the interval dy about y and velocity in the interval du_y about u_y, and n is the number of particles per unit width in the bed. The conservation equation (12) can easily be solved by using the characteristic variables u_y and $\eta=(u_y^2/2+gy)$; the equation in terms of this characteristic variable becomes

$$\frac{df}{du_y}=0. \qquad (13)$$

Equation (13) indicates that f is only a function of η, and this can be solved using the condition for $f(0,u_y)$ from Eq. (10) at $y=0$, to obtain the final expression for the distribution function:

$$f(y,u_y)=\frac{g}{T}\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{u_y^2}{2T}-\frac{gy}{T}\right). \qquad (14)$$

Note that the present definition of the distribution function is slightly different from the one commonly used in kinetic theory of gases. Usually, the distribution function at a height y is defined such that $\rho(y)f(y,u_y)du_y$ is the density of particles with velocities in the interval du_y about u_y at a height y. For a Maxwell-Boltzmann distribution, the density is defined as $\rho(y)=\rho(0)\exp(-gy/T)$, and the distribution function is defined as $f(y,u_y)=(1/\sqrt{2\pi T})\exp(-u_y^2/2T)$. In the present analysis, it is more convenient to use the above definition, where $f(y,u_y)dy\,du_y$ is the number of particles in the interval dy about y in real space and in du_y about u_y in velocity space.

For an *asymmetric* amplitude function for the vibrating surface, the mean velocity $\langle U\rangle_S$ is not zero, and the solution for the conservation equation (8) assumes a different form. It can be seen from Eq. (8) that the particle velocity u_y scales as $(\langle U\rangle_S/\epsilon_I)$, in contrast to the scaling $u_y\sim(\langle U^2\rangle_S/\epsilon_I)^{1/2}$ for a symmetric amplitude function. In this case, if $\langle U^2\rangle_S \sim\langle U\rangle_S^2$, the leading order equation for the distribution function is

$$\epsilon_I\left(2Fu_y+u_y^2\frac{dF}{du_y}\right)-2\langle U\rangle_S\left(u_y\frac{dF}{du_y}+F\right)=0. \qquad (15)$$

It can easily be verified that the solution for the above equation is a δ function,

$$F(u_y)=\delta\left(u_y-\frac{2\langle U\rangle_S}{\epsilon_I}\right). \qquad (16)$$

However, in the vicinity of $u_y=(2\langle U\rangle_S/\epsilon_I)$, the gradient of distribution function becomes large, and the higher order derivatives in the conservation equation could become significant. It turns out that the width of this region is $O(\langle U\rangle_S/\epsilon_I^{1/2})$, and the behavior in this region is determined using the substitution $u_y=(\langle U\rangle_S/\epsilon_I)(2+\epsilon_I^{1/2}v)$, where v is $O(1)$. The leading order conservation equation for v, which represents the deviation of u_y from $2\langle U\rangle_S/\epsilon_I^{1/2}$, is

$$2\left(\frac{\langle U^2\rangle_S}{\langle U\rangle_S^2}-1\right)\frac{d^2F}{dv^2}+\epsilon_I^2\left(v\frac{dF}{dv}+F\right)=0. \qquad (17)$$

The solution of the above equation is, once again, a Gaussian distribution

$$F(v)=\sqrt{\frac{2}{\pi T_{ia}}}\exp\left(-\frac{v^2}{2T_{ia}}\right), \qquad (18)$$

where

$$T_{ia}=\frac{2}{\epsilon_I^2}\left(\frac{\langle U^2\rangle_S}{\langle U\rangle_S^2}-1\right). \qquad (19)$$

The above distribution $F(v)$ is valid for $u_y>0$, and the distribution function for $u_y<0$ is an image of this about the $u_y=0$ axis. Therefore, the distribution function at the vibrating surface is a bimodal distribution consisting of two Gaussian distributions centered at $\pm 2\langle U\rangle_S/\epsilon_I$.

It is interesting to note that the above result predicts $T_{ia}=0$ for $\langle U^2\rangle_S=\langle U\rangle_S^2$. This corresponds to the case where the amplitude of the surface $A(t)$ is a "sawtooth" function $A(t)=A_0[\omega t-\mathrm{mod}(\omega t)]$ where $\mathrm{mod}(\omega t)$ is the highest integer less than ωt, and the velocity has a constant value $A_0\omega$. In this case, it can easily be verified from Eq. (1) that the velocity at steady state, u_y, also has a constant value $u_y=U(1+e)/(1-e)$, and the exact solution for the distribution function is

$$F(u_y)=\delta\left(u_y-\frac{U(1+e)}{1-e}\right). \qquad (20)$$

The above analysis indicates that the distribution function at the vibrating surface is bimodal, with two peaks at $[\pm\langle U\rangle_S/(1-e)]$, and the width of each of these peaks is $O[\langle U\rangle_S/(1-e)^{1/2}]$. The form of the distribution function at any height can easily be determined using the method of characteristics used earlier. If the distribution function at the vibrating surface is given by the leading approximation (16), the distribution function at any height y is

$$f(y,u_y)=\left(\frac{4\langle U\rangle_S}{\epsilon_I g}\right)^{-1}\delta\left(u_y\pm\sqrt{\frac{2\langle U\rangle_S}{\epsilon_I}-2gy}\right) \qquad (21)$$

for $0\leqslant y\leqslant 2\langle U\rangle_S/\epsilon_I g$.

The distribution function (18) and (19) was also verified using computer simulations similar to those used for the symmetric amplitude function. In these simulations, the velocity of the vibrating surface U was set equal to a random number x in the interval $0\leqslant x\leqslant 1$, so that $\langle U\rangle_S\neq 0$ in this case. The results are shown in Fig. 2, and it is seen that the

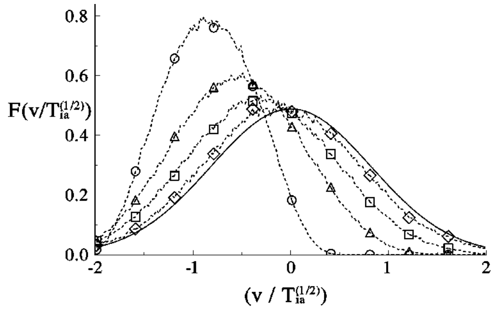

FIG. 2. Velocity distribution $F(v/\sqrt{T_{ia}})$ as a function of $v/\sqrt{T_{ia}}$ for a single particle colliding with an vibrating surface with an asymmetric amplitude function, where dissipation is due to inelasticity. Here, T_{ia} is given by Eq. (19). Solid line, analytical result (18); ○, $\epsilon_I=0.7$; △, $\epsilon_I=0.3$; □, $\epsilon_I=10^{-1}$; ◇, $\epsilon_I=10^{-2}$.

agreement is excellent for $\epsilon_I=10^{-2}$ and 10^{-1}, and fairly good for $\epsilon_I=0.3$. There is a significant quantitative difference between the asymptotic and numerical results for $\epsilon_I=0.7$, but the qualitative nature of the distribution function is well captured by the asymptotic result.

B. Dissipation due to fluid drag

The configuration and the coordinate system used for the present case is identical to that in the preceding subsection. However, the dissipation in the present case is due to the drag acting on the particle between successive collisions, and the coefficient of restitution for a particle collision with the vibrating surface is set equal to 1. The acceleration of the particle between successive collisions is considered to be of the form

$$\frac{du_y}{dt}=-g-\mu u_y, \tag{22}$$

where the drag force is considered to be linear in the velocity and μ is the ratio of the drag coefficient and the mass of a particle and has units of inverse time. The particle velocity is large compared to the velocity of the vibrating surface in the limit $(\mu U/g)\ll 1$ considered here, where U is the magnitude of the velocity of the vibrating surface. In this limit, the change in the particle velocity between successive collisions due to drag is small compared to the velocity of a particle.

The position and the velocity of a particle between successive collisions can be determined by solving the dynamical equation (22),

$$u_y=u_y'\exp(-\mu t)-\frac{g}{\mu}[1-\exp(-\mu t)], \tag{23}$$

$$y=\frac{u_y'}{\mu}[1-\exp(-\mu t)]-\frac{gt}{\mu}+\frac{g}{\mu^2}[1-\exp(-\mu t)], \tag{24}$$

where u_y' is the velocity after a collision with the vibrating surface. The velocity of the particle before the subsequent collision, u_y'', is determined by first calculating the time required from the condition $y=0$, and inserting this into the expression (23) for the velocity. The expression for the velocity u_y'' cannot be determined explicitly, but the velocity correct to second order in the drag coefficient μ in the small dissipation limit is

$$u_y''=-u_y'+\frac{2\mu u_y'^2}{3g}-\frac{4\mu^2u_y'^3}{9g^2}. \tag{25}$$

The velocity of the particle after the collision with the surface, u_y, is related to u_y' and u_y'' by

$$\begin{aligned}u_y-U'&=-(u_y''-U')\\&=u_y'-\frac{2\mu u_y'^2}{3g}+\frac{4\mu^2u_y'^3}{9g^2}+U'.\end{aligned} \tag{26}$$

The method used to determine the distribution function in the present case is identical to that for a system with dissipation due to inelastic collisions. The average flux of particles entering a differential volume du_y about u_y correct to second order in the small parameter μ, analogous to Eq. (3), is

$$N_{\rm in}(u_y)du_y=\left\langle\left(u_y'-\frac{2\mu u_y'^2}{3g}+\frac{4\mu u_y'^3}{9g^2}+U'\right)F(u_y')du_y'\right\rangle_S, \tag{27}$$

where u_y' is related to u_y by Eq. (26), and the relative velocity in Eq. (27) is different from that in Eq. (3) because a particle which leaves the surface with a velocity u_y' has a velocity $-u_y'+(2\mu u_y'^2/3g)-(4\mu u_y'^3/9g^2)$ before the subsequent collision with the surface. The average flux of particles leaving the differential volume du_y about u_y, analogous to Eq. (6), is

$$N_{\rm out}(u_y)du_y=\left\langle\left(u_y-\frac{2\mu u_y^2}{3g}+\frac{4\mu u_y^3}{9g^2}+U\right)F(u_y)du_y\right\rangle_S, \tag{28}$$

where the relative velocity is once again different from that in Eq. (6) because a particle which leaves the surface with velocity u_y after a collision has a velocity $-u_y+(2\mu u_y^2/3g)-(4\mu u_y^3/9g^2)$ before the subsequent collision. The velocity distribution function $F(u_y')$ is expressed using a Taylor series expansion in the parameter $u_y'-u_y$ as before, and the terms proportional to μ, $\langle U\rangle_S$, and $\langle U^2\rangle_S$ are retained to give

$$\begin{aligned}&\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2}+\frac{dF}{du_y}\right)-\langle U\rangle_S\left(u_y\frac{dF}{du_y}+F\right)\\&\quad+\frac{\mu}{g}\left(\frac{u_y^3}{3}\frac{dF}{du_y}+u_y^2F\right)=0.\end{aligned} \tag{29}$$

Equation (29) is the conservation equation for the distribution function when the dissipation is due to drag.

For a vibrating surface with a symmetric amplitude function, $\langle U\rangle_S=0$, and Eq. (29) reduces to

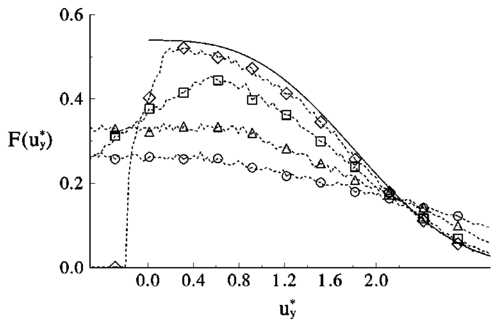

FIG. 3. Velocity distribution $F(u_y^*)$ as a function of $u_y^* = u_y/(\langle U^2\rangle_S g/\mu)^{1/3}$ for a single particle colliding with a vibrating surface with a symmetric amplitude function, where dissipation is due to viscous drag. Solid line, solution of Eq. (31); ○, $(\langle U^2\rangle_S \mu^2/g^2)^{1/3}=0.7$; △, $(\langle U^2\rangle_S \mu^2/g^2)^{1/3}=0.3$; □, $(\langle U^2\rangle_S \mu^2/g^2)^{1/3}=10^{-1}$; $(\langle U^2\rangle_S \mu^2/g^2)^{1/3}=10^{-2}$.

$$\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2}+\frac{dF}{du_y}\right)+\frac{\mu}{g}\left(\frac{u_y^3}{3}\frac{dF}{du_y}+u_y^2F\right)=0. \quad (30)$$

This equation cannot be solved analytically, but a numerical solution can be obtained when the equation is expressed in terms of a dimensionless variable $u_y^* = u_y/(\langle U^2\rangle_S g/\mu)^{1/3}$,

$$u_y^*\frac{d^2F}{du_y^{*2}}+\left(1+\frac{u_y^{*3}}{3}\right)\frac{dF}{du_y^*}+u_y^{*2}F=0. \quad (31)$$

The above equation has two linearly independent solutions. In the limit $u_y^*\to 0$, one of the solutions has the behavior $F_1(u_y^*)=1-u_y^{*3}/9$, and the second has a divergence $F_2(u_y^*)\sim \ln(u_y^*)$. In the limit $u_y^*\to\infty$, the two solutions have the limiting behavior $F_1(u_y^*)\sim\int_{u_y^*}^{\infty}\exp(-u_y^{*3}/9)$, while the second solution has the behavior $F_2\sim u_y^{*-3}$. Since the solution F_2 is divergent at $u_y^*=0$, this is neglected and the solution for the distribution function is determined by numerically integrating Eq. (31) using the condition $F=1$ at $u_y^*=0$. The resulting function was then normalized so that $\int du_y F(u_y)=1$.

The above results indicate that the mean square velocity of the particle scales as $(\langle U^2\rangle_S g/\mu)^{2/3}$, and the distribution function is very different from the Gaussian distribution for the case where dissipation is due to inelastic collisions. The numerical solution for the conservation equation (31) is compared with the computer simulations in Fig. 3. The procedure used for the simulation is identical to that used for dissipation due to inelastic collisions in the preceding section, but the velocity of the particles between successive collisions is updated using Eqs. (23) and (24). Figure 3 shows excellent agreement between the analytical predictions and simulation results as a function of the dimensionless parameter $(\langle U^2\rangle_S\mu^2/g^2)^{1/3}$. It is seen that as in the case of dissipation due to inelastic collisions, there is excellent agreement for $(\langle U^2\rangle_S\mu^2/g^2)^{1/3}=10^{-2}$ and 10^{-1}, and reasonable agreement for $\langle U^2\rangle_S\mu^2/g^2)^{1/3}=0.3$. The agreement is not very good for $\langle U^2\rangle_S\mu^2/g^2)^{1/3}=0.7$, but the qualitative features of the distribution function are captured by the asymptotic analysis. In addition, there is some discrepancy between the analytical and simulation results near $u_y^*=0$ for reasons discussed in the preceding subsection.

For a distribution function with an *asymmetric* amplitude function, $\langle U\rangle_S\neq 0$, and it can be seen from Eq. (29) that $u_y \sim(\langle U\rangle_S g/\mu)^{1/2}$. If $\langle U^2\rangle_S\sim\langle U\rangle_S^2$, then the leading order conservation equation is

$$-\langle U\rangle_S\left(u_y\frac{dF}{du_y}+F\right)+\frac{\mu}{g}\left(\frac{u_y^3}{3}\frac{dF}{du_y}+u_y^2F\right)=0. \quad (32)$$

It can easily be verified that the solution of the above equation is a δ function,

$$F(u_y)=\delta\left(u_y-\sqrt{\frac{3\langle U\rangle_S g}{\mu}}\right). \quad (33)$$

However, as in the preceding section, the higher gradients become significant in the region near $u_y=(3\langle U\rangle_S g/\mu)^{1/2}$, and it is necessary to use an expansion similar to that used in the preceding subsection. It is useful to express the results in terms of a dimensionless parameter $\epsilon_D=(\langle U\rangle_S\mu/g)^{1/2}$. The expansion for the velocity u_y, in terms of this parameter, is

$$u_y=\left(\frac{\langle U\rangle_S g}{\mu}\right)^{1/2}(\sqrt{3}+\sqrt{\epsilon_D}w), \quad (34)$$

where w is $O(1)$. The conservation equation, expressed in terms of the variable w, is

$$\sqrt{3}\left(\frac{\langle U^2\rangle_S}{\langle U\rangle_S^2}-1\right)\frac{d^2F}{dw^2}+2\epsilon_D^2\left(w\frac{dF}{dw}+F\right)=0. \quad (35)$$

The solution of the above equation is a Gaussian distribution

$$F=\sqrt{\frac{2}{\pi T_{va}}}\exp\left(\frac{-w^2}{2T_{va}}\right), \quad (36)$$

where

$$T_{va}=\frac{\sqrt{3}(\langle U^2\rangle_S-\langle U\rangle_S^2)}{2\epsilon_D^2\langle U\rangle_S^2}. \quad (37)$$

The velocity distribution function for the present case is sharply peaked at $\pm(3\langle U\rangle_S g/\mu)^{1/2}$ at the vibrating surface, and the distribution function is $O(1)$ for velocities $O(\epsilon_D^{1/2}(\langle U\rangle_S g/\mu))^{1/2}$ different from these peak positions. The distribution function (36) is valid for $u_y>0$, and the leading order distribution for $u_y<0$ is a mirror image of this about the $u_y=0$ axis. Therefore, the distribution function at the vibrating surface is a bimodal distribution consisting of two Gaussian functions centered at $\pm(3\langle U\rangle_S g/\mu)^{1/2}$.

The result (37) indicates that $T_{va}=0$ as $\langle U^2\rangle_S=\langle U\rangle_S^2$, which corresponds to a ''sawtooth'' for the amplitude function as explained at the end of the preceding subsection. In this case, the velocity of the surface U is a constant, and the velocity of the particle can be directly determined from Eq. (26). Correct to leading order in the drag coefficient μ, the velocity of the particle is $u_y=(3\langle U\rangle_S g/\mu)^{1/2}$, though there are higher order terms which can be determined using a Tay-

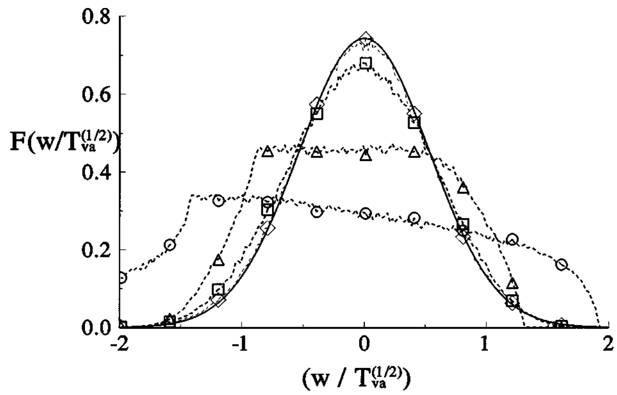

FIG. 4. Velocity distribution $F(w/\sqrt{T_{va}})$ as a function of $w/\sqrt{T_{va}}$ for a single particle colliding with an vibrating surface with an asymmetric amplitude function, where dissipation is due to viscous drag. Here, T_{va} is given by Eq. (37). Solid line, analytical result (36); ○, $\epsilon_D=0.7$; △, $\epsilon_D=0.3$; □, $\epsilon_D=10^{-1}$; ◇, $\epsilon_D=10^{-2}$.

lor series expansion of Eqs. (23) and (24). The distribution function is a δ function in this case as well.

The analytical predictions are compared with simulation results in Fig. 4. The simulation procedure is identical to that used for an inelastic particle colliding with a vibrating surface with an asymmetric wave function, and the change in velocity between successive collisions is determined using Eq. (22). The results indicate that the analysis correctly captures the distribution function for $\epsilon_D=10^{-2}$ and 10^{-1}, but there is some difference between analytical and simulation results for $\epsilon_D=0.3$ and $\epsilon_D=0.7$.

The above analysis can easily be generalized to include more complicated forms of the drag force. For example, consider a drag law of the form

$$\frac{du_y}{dt}=-g-\mu_m u_y|u_y|^{m-1} \tag{38}$$

and the generalized distribution functions are briefly presented here. If the velocity of a particle after a collision is u_y', the velocity before the subsequent collision in the small dissipation limit is [equivalent of Eq. (25)]

$$u_y''=-u_y'+\frac{2\mu_m u_y'^{\,m+1}}{(m+2)g}. \tag{39}$$

This can be used to determine the conservation equation for the distribution function [equivalent to Eq. (29)]

$$\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2}+\frac{dF}{du_y}\right)-\langle U\rangle_S\left(u_y\frac{dF}{du_y}+F\right)+\frac{\mu_m}{g}\left(\frac{u_y^{m+2}}{(m+2)}\frac{dF}{du_y}+u_y^{m+1}F\right)=0. \tag{40}$$

For a vibrating surface with a symmetric amplitude function $\langle U\rangle_S=0$, the distribution function can be expressed in terms of a dimensionless variable $u_y^*=u_y/(\langle U^2\rangle_S g/\mu_m)^{1/(m+2)}$,

$$u_y^*\frac{d^2F}{du_y^{*2}}+\left(1+\frac{u_y^{*m+2}}{(m+2)}\right)\frac{dF}{du_y^*}+u_y^{*m+1}F=0. \tag{41}$$

The above equation also has to be solved numerically, but the form of the equation indicates that the velocity $u_y\sim(\langle U^2\rangle_S g/\mu_m)^{1/(m+2)}$.

For a vibrating surface with an asymmetric amplitude function, the equivalent of Eq. (32) is

$$-\langle U\rangle_S\left(u_y\frac{dF}{du_y}+F\right)+\frac{\mu_m}{g}\left(\frac{u_y^{m+2}}{(m+2)}\frac{dF}{du_y}+u_y^{m+1}F\right)=0. \tag{42}$$

From the above equation, it can easily be verified that the leading order distribution function [analogous to Eq. (33)] is

$$F(u_y)=\delta\left(\frac{\langle U\rangle_S g(m+2)}{\mu_m}\right)^{1/(m+1)}. \tag{43}$$

The distribution function near $u_y=[\langle U\rangle_S g(m+2)/\mu_m]^{1/(m+1)}$ is determined using the scaled variable w defined as [equivalent of Eq. (34)]

$$w=\left(\frac{\langle U\rangle_S g}{\mu_m}\right)^{1/(m+1)}[(m+2)^{1/(m+1)}+\sqrt{\epsilon_D}w], \tag{44}$$

where the parameter $\epsilon_D=(\langle U\rangle_S^m\mu_m/g)^{1/(m+1)}$. The solution for the distribution function in terms of this scaled variable is a Gaussian distribution given by Eq. (36), where T_{va} is [equivalent of Eq. (37))]

$$T_{va}=\frac{(m+2)^{1/(m+1)}(\langle U^2\rangle_S-\langle U\rangle_S^2)}{2\epsilon_D^2\langle U\rangle_S^2}. \tag{45}$$

III. FIRST CORRECTION DUE TO BINARY COLLISIONS

The distribution function that includes the leading order effect of binary collisions is derived in the limit where the binary collisions are sufficiently infrequent compared to particle collisions with the vibrating surface, and the distribution function approaches the single particle distribution function between successive binary collisions. It is shown a little later that this limit is observed in the parameter regime $\delta_I\ll 1$, where $\delta_I=2nr/(1-e)$, n is the number of particles per unit width and r is the particle radius. This correction to the distribution function at the vibrating surface is derived self-consistently at steady state by equating the collisional accumulation and depletion of particles in a differential volume in velocity space. The distribution function as a function of height is then obtained using the method of characteristics used in the preceding section. Since most of the particles have distribution functions that are close to the single particle distribution function in the absence of binary collisions, the leading effect of binary collisions is due to collisions between two particles having a velocity distribution given by Eq. (14). It is shown a little later, after deriving the distribution function, that the error in the collisional flux due to this approximation is $O(\delta_I)$ smaller than the leading order collisional flux.

Before proceeding to derive the correction to the distribution function, it is necessary to determine the change in velocity due to a binary collision. Consider a collision between two particles with initial velocities $(0,u'_y)$ and $(0,u'_{1y})$ in the interval $dy^\dagger$ about the vertical position $y^\dagger$ such that one of the particles has a final velocity $(u_x^\dagger,u_y^\dagger)$, and that the line joining the centers of the particles at the point of collision makes an angle θ with the horizontal. If the collision is elastic, the initial and final velocities are related by

$$u_x^\dagger = -\frac{w'_y}{2}\sin(2\theta),$$
$$u_y^\dagger = v'_y + \frac{w'_y}{2}\cos(2\theta), \tag{46}$$

where $v'_y=(u'_y+u'_{1y})/2$ and $w'_y=(u'_y-u'_{1y})$. The above equations can be inverted to express v'_y and w'_y in terms of $u_x^\dagger$ and $u_y^\dagger$:

$$v'_y = u_y^\dagger + u_x^\dagger \cot(2\theta),$$
$$w'_y = -2u_x^\dagger \csc(2\theta). \tag{47}$$

The collision described above results in an accumulation of particles in the differential volume $dy^\dagger du_x^\dagger du_y^\dagger$ about $(y^\dagger,u_x^\dagger,u_y^\dagger)$. The rate of collisions per unit length ℓ in the horizontal direction leading to accumulation is

$$(\mathcal{R}/\ell) = n^2 dy^\dagger dv'_y dw'_y \int d\mathbf{k}\, f_0(y^\dagger,v'_y) f_0(y^\dagger,w'_y) \times (2r\mathbf{w}'\cdot\mathbf{k}) du_x^\dagger du_y^\dagger, \tag{48}$$

where $\mathbf{k}$ is the unit vector in the direction of the line joining the centers of the two particles, and is directed from the center of the particle with velocity $(0,u'_y)$ to the center of the particle with velocity $(0,u'_{1y})$. The distribution functions $f_0(y^\dagger,v'_y)$ and $f_0(y^\dagger,w'_y)$ [from Eq. (14)] are

$$f_0(y^\dagger,v'_y) = \frac{g}{T}\frac{1}{\sqrt{\pi T}}\exp\left(-\frac{v_y'^2}{T}-\frac{gy^\dagger}{T}\right) f_0(y^\dagger,w'_y) = \frac{g}{T}\frac{1}{\sqrt{4\pi T}}\exp\left(-\frac{w_y'^2}{4T}-\frac{gy^\dagger}{T}\right). \tag{49}$$

In addition, the integral in Eq. (48) is carried out only for $\mathbf{w}'\cdot\mathbf{k}\geqslant 0$, since the particles move away from each other for $\mathbf{w}'\cdot\mathbf{k}<0$. In deriving Eq. (48), the distribution function $f(y^\dagger,u_x^\dagger,u_y^\dagger)$ is assumed to be the distribution function $f_0(y^\dagger,u'_y)\,\delta(u'_x)=(2\pi T)^{-1/2}\exp(-u_y'^2/2T)\delta(u'_x)$ in the absence of binary collisions. The justification for this assumption is as follows. It is shown a little later that the fraction of particles with velocities $O(\sqrt{T})$ different from $u_x^\dagger=0$ is $O(\delta_I)$, and in the limit $\delta_I\ll 1$ most of the particles have velocities that are $O(\delta_I)$ different from $u_x^\dagger=0$. Consequently, the error made due to this assumption is also $O(\delta_I)$ smaller than the collisional flux of particles.

Using the relations (47) between the velocities before and after collision, the rate of binary collisions between particles in the interval $dy^\dagger$ about the position $y^\dagger$ which lead to an accumulation of particles in the differential volume $du_x^\dagger du_y^\dagger$ about $(u_x^\dagger,u_y^\dagger)$ is

$$\begin{aligned}\mathcal{R}/\ell &= 8n^2 r u_x^\dagger \int d\theta\, f_0(y^\dagger,v'_y) f_0(y^\dagger,w'_y)(-\csc(2\theta)^2) \times \sin(\theta) du_x^\dagger du_y^\dagger dy^\dagger \\ &= \frac{\sqrt{2}rn^2g^2}{\pi T^3}\left[\exp\left(-\frac{u_x^{\dagger 2}}{T}-\frac{u_y^{\dagger 2}}{2T}-\frac{2gy^\dagger}{T}\right) \times \int d\psi\,\exp\left(-\frac{\psi^2}{T}\right) S(\psi,u_x^\dagger,u_y^\dagger)\right] du_x^\dagger du_y^\dagger dy^\dagger,\end{aligned} \tag{50}$$

$$S(\psi,u_x^\dagger,u_y^\dagger) = \left[\frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{1}{1+\cot(2\theta)^2}}\right]^{1/2}. \tag{51}$$

In deriving Eq. (50), the relation between the differential volumes in velocity space $dv'_y dw'_y = 2\csc(2\theta) du_x^\dagger du_y^\dagger$ has been used, and the transformation

$$\psi = \left(\sqrt{2}u_x^\dagger \cot(2\theta) + \frac{u_y^\dagger}{\sqrt{2}}\right) \tag{52}$$

has been employed to convert the independent variable from the angle θ to the variable ψ. It can easily be verified that the limits of integration for ψ are $-\infty\leqslant\psi\leqslant\infty$. $S(\psi,u_x^\dagger,u_y^\dagger)$ is the factor $\sin(\theta)$ in Eq. (50) written in terms of ψ, $u_x^\dagger$, and $u_y^\dagger$.

The above collisions at different vertical positions $y^\dagger$ in the bed lead to an accumulation of particles in the differential volume $du_x du_y$ about (u_x,u_y) *at the vibrating surface*, where (u_x,u_y) are related to $(u_x^\dagger,u_y^\dagger)$ as follows:

$$u_x^\dagger = u_x, \quad \frac{u_y^2}{2} = \frac{u_y^{\dagger 2}}{2} + gy^\dagger. \tag{53}$$

The total rate of accumulation of particles at the vibrating surface due to binary collisions is given by an integral over the distance from the vibrating surface $y^\dagger$ of the collisional flux that leads to an accumulation in the differential volume $du_x du_y$ about (u_x,u_y),

$$\begin{aligned}N_{\text{in}}^{(b)}(0,u_x,u_y)du_x du_y &= \frac{\sqrt{2}rn^2g^2}{\pi T^3} du_x^\dagger du_y^\dagger \left[\exp\left(-\frac{u_x^2}{T}\right) \times \int dy^\dagger \exp\left(-\frac{u_y^{\dagger 2}}{2T}-\frac{2gy^\dagger}{T}\right) \times \int d\psi\,\exp\left(-\frac{\psi^2}{T}\right) S(\psi,u_x^\dagger,u_y^\dagger)\right].\end{aligned} \tag{54}$$

It is convenient to represent the right side of Eq. (54) using the independent variables $(u_y,u_y^\dagger)$ instead of $(u_y^\dagger,y^\dagger)$. It can easily be verified that the differential volumes in these two coordinates are related by $du_y^\dagger dy^\dagger = (|u_y|/g) du_y du_y^\dagger$, and Eq. (54) then becomes

$$N_{\rm in}^{(b)}(0,u_x,u_y) = \frac{\sqrt{2} r n^2 g}{\pi T^3}\left[\exp\left(-\frac{u_x^2+u_y^2}{T}\right)|u_y| \times \int_{u_y}^{(u_y^2)^{1/2}} du_y^\dagger \exp\left(\frac{u_y^{\dagger 2}}{2T}\right) \times \int d\psi \exp\left(-\frac{\psi^2}{T}\right) S(\psi, u_x^\dagger, u_y^\dagger)\right]. \tag{55}$$

In the above expression, the upper and lower limits of integration for the variable $u_y^\dagger$ have been identified so that the position $y^\dagger \geqslant 0$ ($u_y^{\dagger 2} \leqslant u_y^2$), and the condition that the time taken for a particle with initial velocity $u_y^\dagger$ to attain a final velocity u_y under the influence of gravity is positive semidefinite ($u_y^\dagger \geqslant u_y$). Note that the integral in Eq. (55) is nonzero only for $u_y < 0$, because binary collisions only cause an accumulation of downward traveling particles at the vibrating surface.

There is also a depletion of particles in the differential volume $(du_x^\dagger du_y^\dagger)$ about $(u_x^\dagger, u_y^\dagger)$ in velocity space due to binary collisions. The rate of binary collisions in the interval $dy^\dagger$ about $y^\dagger$ which lead to a depletion of particles in the differential volume $(du_x^\dagger du_y^\dagger)$ about $(u_x^\dagger, u_y^\dagger)$ is calculated using an approximate expression of the form:

$$\mathcal{R}/\ell = 2 r n^2 \int d\mathbf{k} \int du_y' f(0,u_x^\dagger,u_y^\dagger) \frac{1}{\sqrt{2\pi T}} \frac{g}{T} \times \exp\left(\frac{-u_y'^2}{2T}\right) \exp\left(-\frac{2 g y^\dagger}{T}\right) w_y k_y du_x^\dagger du_y^\dagger dy^\dagger, \tag{56}$$

where $\mathbf{k}$ is the unit vector in the direction of the line joining the centers of the two particles at the point of collision, $w_y = u_y^\dagger - u_y'$ is the relative velocity, and the integral is carried out for $w_y k_y > 0$. In deriving the above equation, two assumptions have been made.

(i) The distribution function $f(y^\dagger, u_x', u_y')$ for the particle with velocity (u_x', u_y') involved in the collision has been approximated by the single particle velocity distribution $(2\pi T)^{-1/2} \exp(-u_y'^2/2T)\delta(u_x')\exp(-g y^\dagger/T)$. This approximation is valid when the distribution function of this particle is close to that derived in the absence of binary collisions (14). It is shown a little later, after deriving the distribution function, that the number of particles with horizontal velocities $O(\sqrt{T})$ different from $u_x = 0$ is $O(\delta_I)$ smaller than the number of particles with velocities $O(\delta_I \sqrt{T})$ different from $u_x = 0$. Consequently, the error incurred in the estimation of the rate of collisions due to this approximation is $O(\delta_I)$ smaller than the rate of collisions.

(ii) The relative velocity $[(u_x^\dagger - u_x')^2 + (u_y^\dagger - u_y')^2]^{1/2}$ has been approximated by $|(u_y^\dagger - u_y)|$. The error made due to this approximation is δ_I for particles with horizontal velocities $O(\delta_I \sqrt{T})$ different from $u_x = 0$. However, if a particle has horizontal velocity $O(\sqrt{T})$ different from $u_x^\dagger = 0$, the error made in the rate of binary collisions is of the same magnitude as the rate of binary collisions. However, for such particles, it turns out (as discussed a little later) that the rate of transport of particles in velocity space due to binary collisions is $O(\delta_I)$ smaller than the rate of transport due to collisions with the vibrating surface. Consequently, the above approximation has a maximum error of $O(\delta_I)$ in the estimation of the total rate of transport of particles in velocity space.

The above discussion indicates that the above approximation for the rate of binary collisions provides a uniform approximation which results in a maximum error of $O(\delta_I)$ in the estimation of the total rate of transport of particles in velocity space.

The collisions between particles with velocities $(u_x^\dagger, u_y^\dagger)$ and (u_x', u_y') at a height $y^\dagger$ result in a depletion of particles with velocity (u_x, u_y) at the vibrating surface, where $(u_x^\dagger, u_y^\dagger)$ are related to (u_x, u_y) by Eq. (53). The contribution of the collisional depletion of particles to the distribution function can be obtained in a manner similar to Eq. (55) for the collisional accumulation:

$$\begin{aligned} N_{\rm out}^{(b)}(0,u_x,u_y) &= \frac{4 r n^2}{T} f(0,u_x,u_y)|u_y| \exp\left(\frac{-u_y^2}{2T}\right) \times \int_{u_y}^{(u_y^2)^{1/2}} du_y^\dagger \exp\left(\frac{u_y^{\dagger 2}}{2T}\right)\left[u_y^\dagger \,\mathrm{erf}\left(\frac{u_y^\dagger}{\sqrt{2T}}\right) + \sqrt{\frac{2T}{\pi}} \exp\left(\frac{-u_y^{\dagger 2}}{2T}\right)\right] \\ &= 8 r n^2 f(0,u_x,u_y)|u_y| \,\mathrm{erf}\left(\frac{|u_y|}{\sqrt{2T}}\right). \end{aligned} \tag{57}$$

In deriving the above relation, the approximation $f(y^\dagger, u_x^\dagger, u_y^\dagger) = f(0,u_x,u_y)\exp(-g y^\dagger/T)$ has been used; the justification for this approximation is identical to the second justification provided after Eq. (56).

The flux in velocity space due to particle collisions with the vibrating surface is calculated next using a procedure similar to that for the conservation equation for the distribution function (14). If the coefficient of restitution is e in the normal direction and e_t in the tangential direction, the fluxes of particles incident on and reflected from the vibrating surface, analogous to Eqs. (4) and (6), are

$$N_{\rm in}^{(v)}(0,u_x,u_y) du_x du_y = n(U + u_y') f(0,u_x',u_y') du_x' du_y', \tag{58}$$

$$N_{\rm out}^{(v)}(0,u_x,u_y) du_x du_y = (u_y + U) f(0,u_x,u_y) du_x du_y. \tag{59}$$

Using the relation (1) between u_y and u_y', and the relation $u_x = e_t u_x'$, the change in the distribution function due to a particle collision with the vibrating surface is derived in a manner similar to Eq. (14) for the single particle distribution function. The velocities (u_x', u_y') are expressed in terms of (u_x, u_y) using the laws for an inelastic collision at the vibrating surface, and a relation between the distribution functions $f(0,u_x',u_y')$ and $f(0,u_x,u_y)$ is obtained in the limit $(u_y'-u_y)\ll u_y$,

$$n[(u_y'+U')f(0,u_x',u_y')-(u_y+U)f(0,u_x,u_y)]$$
$$=n\epsilon_I[T(u_y\partial_{u_y}^2+\partial_{u_y})+u_y^2\partial_{u_y}+(2+a_t)u_y$$
$$+a_tu_yu_x\partial_{u_x}]f(0,u_x,u_y), \tag{60}$$

where $a_t=(1-e_t)/(1-e)$,

$$N_{\rm in}^{(v)}(0,u_x,u_y)-N_{\rm out}^{(v)}(0,u_y,u_y)$$
$$=n\epsilon_I[T(u_y\partial_{u_y}^2+\partial_{u_y})u_y^2\partial_{u_y}+(2+a_t)u_y$$
$$+a_tu_xu_y\partial_{u_x}]f^*(0,u_x,u_y). \tag{61}$$

The equation for the distribution function at steady state is obtained using the condition that the net flux

$$N_{\rm in}^{(v)}+N_{\rm in}^{(b)\prime}-N_{\rm out}^{(v)}-N_{\rm out}^{(b)\prime}=0. \tag{62}$$

The flux balance equation for the scaled distribution function, $f^*(0,u_x^*,u_y^*)=(T^2/g)f(0,u_x,u_y)$, in terms of the scaled velocities $u_x^*=(u_x/\sqrt{T})$ and $u_y^*=(u_y/\sqrt{T})$,

$$[u_y^*\partial_{u_y^*}^2+(u_y^{*2}+1)\partial_{u_y^*}+(2+a_t)u_y^*+a_tu_y^*u_x^*\partial_{u_x^*}$$
$$-\delta_Ig_1(u_y^*)]f^*(0,u_x^*,u_y^*)=-\delta_Ig_2(u_x^*,u_y^*), \tag{63}$$

where the parameter $\delta_I=(2rn/\epsilon_I)$, and the functions $g_1(u_y^*)$ and $g_2(u_x^*,u_y^*)$ are

$$g_1(u_y^*)=4u_y^*\,\mathrm{erf}\left(\frac{u_y^*}{\sqrt{2}}\right), \tag{64}$$

$$g_2(u_x^*,u_y^*)=\frac{1}{\sqrt{2}\pi}\Bigg[\exp(-(u_x^{*2}+u_y^{*2}))u_y^*$$
$$\times\int_{-u_y^*}^{u_y^*}dv\,\exp\left(\frac{v^2}{2}\right)\int_{-\infty}^{\infty}d\psi^*$$
$$\times\exp\left(-\frac{\psi^{*2}}{T}\right)S(\psi^*,u_x^*,v)\Bigg], \tag{65}$$

where $\psi^*=(\psi/\sqrt{T})$. Note that the equations for the distribution function (63) [and the equations for the one-dimensional distribution function (9)] are derived for $u_y^*\geqslant 0$. Consequently, the functions $g_1(u_y^*)$ and $g_2(u_y^*)$ refer to the domain $u_y^*\geqslant 0$.

Equation (63) for the distribution function has to be solved numerically, but it is necessary to obtain an analytical solution in the limit $u_x\to 0$ using an asymptotic expansion in the small parameter δ_I. Since the distribution function is close to the distribution function in the absence of collisions derived previously, it is expected that the variation of the distribution function along the u_x axis near $u_x=0$ is large compared to that along the u_y axis. In a naive asymptotic expansion, the terms proportional to δ_I are neglected in the conservation equation, and the following leading order solution is obtained for f^*:

$$f^*(0,u_x^*,u_y^*)=\frac{1}{N}\exp\left(-\frac{u_y^{*2}}{2}\right)|u_x^*|^{-1}+O(\delta_I). \tag{66}$$

This solution is not satisfactory, however, because the integral of the distribution function with respect to u_x^* diverges in the limit $u_x^*\to 0$, and the distribution function cannot be normalized. The difficulty is resolved by realizing that the $O(\delta_I)$ correction to the conservation equation (63) could cause a variation of $O(\delta_I)$ in the exponent of $|u_x^*|$ in Eq. (66), and this could render the integral convergent. To incorporate this possibility, the distribution function is written as $f^*(u_x^*,u_y^*)=|u_x^*|^{c\delta_I-1}(f_0^*+\delta_If_1^*)$. The constant c is determined from the solvability condition for the homogeneous part of the conservation equation (63) [without the inhomogeneous term $-\delta_Ig_2(u_x^*,u_y^*)$]. The inhomogeneous part of Eq. (63) causes a correction of $O(\delta_I)$ to the leading order equation, and the particular solution can be determined by numerically integrating Eq. (63). This is carried out a little later.

When the assumed form $f^*(0,u_x^*,u_y^*)=|u_x^*|^{c\delta_I-1}(f_0^*+\delta_If_1^*)$ is inserted into the homogeneous part of Eq. (63), the leading order terms in the equation sum to zero for $f_0^*\propto\exp(-u_y^{*2}/2)$, while the $O(\delta_I)$ contribution to the equation is

$$|u_x^*|^{c\delta_I-1}[u_y^*d_{u_y^*}^2+(u_y^{*2}+1)d_{u_y^*}+2u_y^*]f_1^*(u_y^*)$$
$$=|u_x^*|^{c\delta_I-1}[g_1(u_y^*)-a_tcu_y^*]f_0^*(u_y^*). \tag{67}$$

The constant c can be determined from the solvability condition for the above equation as follows. The operator $L[Y(u_y^*)]$ is defined as

$$L[f^*(u_y^*)]\equiv[u_y^*\partial_{u_y^*}^2+(u_y^{*2}+1)\partial_{u_y^*}+2u_y^*]f^*(u_y^*), \tag{68}$$

the inner product $\langle Y,Y'\rangle$ is defined as

$$\langle Y,Y'\rangle=\int_0^\infty du_y^*Y(u_y^*)Y'(u_y^*), \tag{69}$$

and the boundary condition $Y(0)=0$ is imposed without loss of generality. The solvability condition for Eq. (67) then reduces to

$$\langle Y^*,\{[\delta_Ig_1(u_y^*)-\delta_Ia_tcu_y^*]f_0^*(u_y^*)\}\rangle=0, \tag{70}$$

where $Y^*(u_y^*)$ is the solution of the equation $L^*[Y^*(u_y^*)]=0$, where L^* is the adjoint of the operator L. It can be easily verified that

$$L^*[Y^*(u_y^*)]=[u_y^*d_{u_y^*}^2+(2-u_y^{*2})d_{u_y^*}]Y^*(u_y^*) \tag{71}$$

and the boundary conditions at $u_y^*=\infty$ require that the function Y^* diverges slower than $\exp(u_y^{*2}/2)$ in this limit. The only solution of Eq. (71) that satisfies this condition is $Y^*(u_y^*)=\mathrm{const}$, and the solvability condition reduces to

$$\int_0^\infty du_y^*\{[g_2(u_y^*)-a_t c u_y^*]f_0^*(u_y^*)\}=0. \tag{72}$$

The solution of the above equation provides the constant c,

$$c=\frac{2}{a_t\sqrt{\pi}}, \tag{73}$$

and the form of the distribution function that is correct up to leading order in the small parameter δ_I is

$$f^*(0,u_x^*,u_y^*)=\frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{u_y^{*2}}{2}\right)(c\,\delta_I)|u_x^*|^{(c\delta_I-1)}\times\exp(-u_x^{*2}). \tag{74}$$

The distribution function at any height y^* can be derived in a manner similar to Eq. (14),

$$f^*(y^*,u_x^*,u_y^*)=\frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{u_y^{*2}}{2}-y^*\right)\times(c\,\delta_I)|u_x^*|^{(c\delta_I-1)}\exp(-u_x^{*2}). \tag{75}$$

Note that the above distribution satisfies the normalization condition in the leading order approximation. The factor $\exp(-u_x^{*2})$ has been included in the definition of the distribution function due to the presence of the same factor on the right side of the conservation equation (63), and it is expected that the distribution function would decay as $\exp(-u_x^{*2})$ in the limit $u_x^*\gg 1$. The presence of this factor renders the integral of the distribution function convergent in the limit $u_x^*\gg 1$, while leading to an error of $O(\delta_I^2)$ in the limit $u_x^*\sim\delta_I$. The above result indicates that the distribution function diverges at $u_x^*=0$, and the rate of divergence increases in the limit $\delta_I\to 0$. It should also be noted that the distribution function is $O(1)$ for $u_x^*=O(\delta_I\sqrt{T})$, and decreases to $O(\delta_I)$ for $u_x^*=O(\sqrt{T})$. The fraction of particles with velocities $u_x^*=O(1)$ is $O(\delta_I)$, and most of the particles have horizontal velocities $u_x^*=O(\delta_I\sqrt{T})$. However, the particles with velocities $u_x^*=O(1)$ make a significant contribution to the mean square of the velocity moments in the horizontal direction, and so it is necessary to determine the distribution function for $u_x^*=O(\sqrt{T})$ to determine these moments. This is done using expansions in orthogonal polynomials.

The distribution function that includes the first effect of collisions is determined using an expansion of the form

$$f^*(0,u_x^*,u_y^*)=\frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{u_y^{*2}}{2}\right)\exp(-u_x^{*2})\times\left[c\,\delta_I|u_x^*|^{(c\delta_I-1)}+\delta_I\sum_{i=0}^{i_f}\sum_{j=0}^{j_f}h_{ij}H_i(u_x^*)G_j\left(\frac{u_y^*}{\sqrt{2}}\right)\right], \tag{76}$$

where h_{ij} are $O(1)$ coefficients and $H_i(u_x^*)$ are Hermite polynomials. $G_j(u_y^*)$ are orthogonal polynomials of order j (obtained by Gramm-Schmidt orthogonalization) which are defined in the domain $0\leq u_y^*<\infty$, with the weighting function $\exp(-u_y^{*2}/2)$, all set equal to 1 at $u_y^*=0$. It can easily be verified that the above distribution function converges to Eq. (74) in the limit $u_x^*\to 0$, because the first term proportional to $|u_x^*|^{(c\delta_I-1)}$ diverges in this limit, while all other terms in the expansion are $O(\delta_I)$ or smaller.

The distribution function (76) is inserted into the conservation equation (63), and the resulting equation only contains terms that are convergent in the limit $u_x^*\to 0$. There is one further approximation made in order to obtain coefficients h_{ij} that are independent of the parameter δ_I. The distribution function (76) is separated into two components,

$$f_a^*(0,u_x^*,u_y^*)=\frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{u_y^{*2}}{2}\right)\times\exp(-u_x^{*2})c\,\delta_I|u_x^*|^{(c\delta_I-1)}, \tag{77}$$

$$f_b^*(0,u_x^*,u_y^*)=\frac{1}{2\sqrt{2\pi}}\delta_I\exp\left(-\frac{u_y^{*2}}{2}\right)\times\exp(-u_x^{*2})\sum_{i=0}^{i_f}\sum_{j=0}^{j_f}h_{ij}H_i(u_x^*)G_j(u_y^*).$$

When $f_a^*(0,u_x^*,u_y^*)$ is inserted into the left side of the conservation equation (63), the resulting expression is

$$\frac{1}{2\sqrt{2\pi}}\Bigg[a_t u_y^*(c\,\delta_I)[c\,\delta_I|u_x^*|^{(c\delta_I-1)}-2|u_x^*|^{(c\delta_I+1)}]\times\exp(-u_x^{*2})\exp\left(\frac{-u_y^{*2}}{2}\right)-\delta_I g_1(u_y^*)f_a^*(0,u_x^*,u_y^*)\Bigg]. \tag{78}$$

In addition to the terms of $O(\delta_I)$, the term proportional to $\delta_I^2|u_x^*|^{(c\delta_I-1)}$ has also been retained in the above expression because it becomes $O(\delta_I)$ for $u_x^*\sim\delta_I$. To determine the coefficients h_{ij}, the right and left sides of the conservation equation (63) are multiplied by $H_p(u_x^*)G_q(u_y^*)$ and integrated over $-\infty<u_x^*<\infty$ and $0<u_y^*<\infty$ to obtain equations for h_{ij}, and these are solved simultaneously to obtain the coefficients h_{ij}. The equation for $p=0$ and $q=0$ reduces to

$$\frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} du_x^* \exp(-u_x^{*2})\int_0^{\infty} du_y^* \exp\left(\frac{-u_y^{*2}}{2}\right) a_t u_y^*(c\delta_I)\right.$$
$$\times[c\delta_I|u_x^*|^{(c\delta_I-1)}-2|u_x^*|^{(c\delta_I+1)}]$$
$$\left.-\delta_I g_1(u_y^*)(c\delta_I)|u_x^*|^{(c\delta_I-1)}\right]$$
$$=-\delta_I\int_{-\infty}^{\infty} du_x^*\int_0^{\infty} du_y^* g_2(u_x^*,u_y^*). \tag{79}$$

The above expression does not contain any of the coefficients h_{ij}, and is equivalent to a solvability condition for the differential equation (63). It can easily be verified that the above equation is identically satisfied, correct to $O(\delta_I)$, for $g_1(u_y^*)$ and $g_2(u_x^*,u_y^*)$ given by Eqs. (64) and (65).

At this point, it is useful to reexamine the assumptions that were made [after Eq. (56)] in deriving the distribution function. Equation (76) for the distribution function confirms that the first assumption is valid. The change in the distribution function due to binary collisions for a particle with horizontal velocity $\sqrt{T}$ different from $u_x^*=0$ is $O(nrf(0,u_x^*,u_y^*))$, while the change in the distribution function due to collisions with the vibrating surface is $O(\epsilon_I f(0,u_x^*,u_y^*))$ from Eq. (61). Consequently, the change in the distribution function due to binary collisions is $O(\delta_I)$ smaller than that due to collisions with the vibrating surface for particles with horizontal velocities $\sqrt{T}$ different from $u_x^*=0$; this confirms the second assumption.

The coefficient of h_{00} in Eq. (79) is zero, and so this coefficient cannot be determined from the conservation equation (63). This coefficient is determined from the normalization condition for the distribution function as follows. The distribution function at any height y in the vibrated material can be obtained from the distribution function at $y=0$ using the condition (13),

$$f^*(y^*,u_x^*,u_y^*)=f^*(0,u_x^*,\eta^*), \tag{80}$$

where the $\eta^*=(u_y^{*2}+2y^*)^{1/2}$ and the dimensionless distance $y^*=gy/T$. It should be noted that in deriving Eq. (80), only the correction due to particles reflected off the vibrating surface has been included, and the correction due to particles at height y^* which have not yet collided with the vibrating surface after a binary collision has been neglected; it can easily be verified that the error to the distribution function due to this approximation is $O(\epsilon_I)$. The normalization condition is expressed in terms of the parameter η^*,

$$1=\int_0^{\infty} dy^*\int_{-\infty}^{\infty} du_y^*\int_{-\infty}^{\infty} du_x^* f^*(y^*,u_x^*,u_y^*)$$
$$=2\int_0^{\infty} d\eta^*\int_{-\infty}^{\infty} du_x^* f^*(0,u_x^*,\eta^*)\,\eta^{*2}$$
$$=\left[\frac{c\delta_I}{2}\Gamma\left(\frac{c\delta_I}{2}\right)+\delta_I\sum_{j=0}^{j_f} l_j h_{0j}\right]$$
$$=\left[1-\frac{c\delta_I\gamma}{2}+\delta_I\sum_{j=0}^{j_f} l_j h_{0j}\right], \tag{81}$$

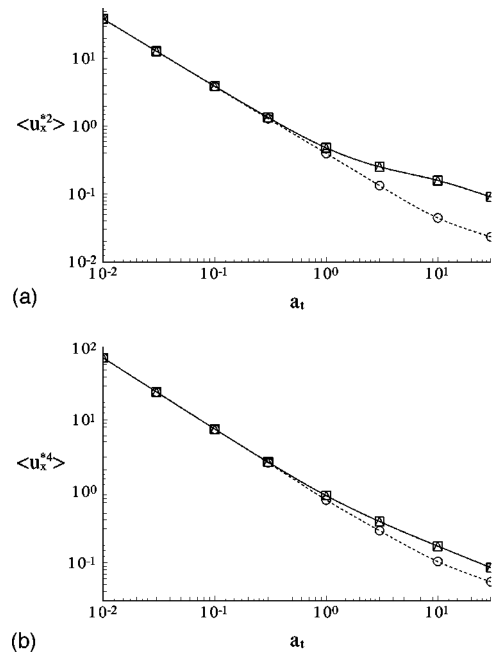

FIG. 5. The $O(\delta_I)$ contribution to the mean square velocity $\langle u_x^{*2}\rangle$ (a) and the $O(\delta_I)$ contribution to the fourth moment of the velocity in the horizontal direction $\langle u_x^{*4}\rangle$ (b) as a function of $a_t=(1-e_t)/(1-e)$ for a system where the dissipation is due to inelastic collisions. ○, $i_f=j_f=3$; △, $i_f=j_f=4$; □, $i_f=j_f=5$.

where $\Gamma(\,)$ is the gamma function, γ is the Euler constant, and the coefficients l_j are

$$l_j=\sqrt{\frac{2}{\pi}}\int_0^{\infty} d\eta^*\ \eta^{*2}\exp\left(-\frac{\eta^{*2}}{2}\right)G_j(\eta^*). \tag{82}$$

The normalization condition (81) gives the expression for the coefficient h_{00}. With this, the calculation of the distribution function that incorporates the first effect of binary collisions is complete.

The moments of the velocity distribution can now be determined using the distribution function (76),

$$\langle\psi(y^*,u_x^*,u_y^*)\rangle$$
$$=\int_{-\infty}^{\infty} du_x^*\int_{-\infty}^{\infty} du_y^*\psi(u_x^*,u_y^*)f^*(y^*,u_x^*,u_y^*). \tag{83}$$

The results of the numerical calculations depend on i_f and j_f, the number of orthogonal polynomials included in the expansions in Eq. (76). However, it can be seen from Fig. 5 that there is very little change in the results for the velocity moments when the number of functions is changed from 4 to 5, and we have assumed that $i_f=5$ and $j_f=5$. The calculations show that the first correction to the second moment in the vertical direction at the vibrating surface is zero because of the normalization condition (81). The first correction to the fourth moment of the velocity distribution in the vertical direction is

$$\langle u_y^{*4}\rangle=(1-3.9975\delta_I). \tag{84}$$

The moments of the horizontal velocity at the vibrating surface are functions of a_t, which is the ratio of the coefficients of restitution in the horizontal and vertical directions. The moments of the horizontal velocity $\langle u_x^{*2}\rangle$ and $\langle u_x^{*4}\rangle$ are shown as a function of a_t in Fig. 5. It is seen that the velocity moments diverge proportional to a_t^{-1} in the limit $a_t\to 0$, because the coefficient of restitution in the horizontal direction tends to 1 in this limit and there is no dampening of the horizontal velocity fluctuations. In the limit $a_t\gg 1$, where the coefficient of restitution in the horizontal direction becomes small, the moments of the horizontal velocity distribution decrease to zero.

The $O(\delta_I)$ contributions to the nontrivial third moments of the velocity distribution, $\langle u_y^{*3}\rangle$ and $\langle u_y^* u_x^{*2}\rangle$, are identically zero because the $O(\delta_I)$ contribution to the distribution function evaluated above is symmetric about $u_y^*=0$. However, there is an $O(nr)$ correction to the third moments due to particles that have not yet collided with the vibrating surface after a binary collision. The contribution to the third moment due to these particles can be evaluated by taking moments of the Boltzmann equation for the velocity distribution. The equations for the third moments, suitably nondimensionalized, are [12]

$$\partial_y n\langle u_y^3\rangle = \frac{\partial_c n\langle u_y^2\rangle}{\partial t}, \tag{85}$$

$$\partial_y n\langle u_y u_x^2\rangle = \frac{\partial_c n\langle u_x^2\rangle}{\partial t}, \tag{86}$$

where $(\partial_c \psi/\partial t)$ is the nondimensionalized rate of change of the moment ψ due to binary collisions between particles. The leading order collision integral can be calculated by considering the collisions between particles traveling in the vertical direction; the error made due to this approximation is $O(\delta_I)$ because the fraction of particles with velocities $O(1)$ different from the terminal velocities is $O(\delta_I)$. With this approximation, the collision integral is

$$\frac{\partial_c \psi}{\partial t} = 2rn^2\int d\mathbf{k}\int_{-\infty}^{\infty} du_y^* \int_{-\infty}^{\infty} du_{1y}^* f_0(y,u_y) f_0(y,u_{1y}) \times[\psi^\dagger(u_y)-\psi(u_y)]\mathbf{w}\cdot\mathbf{k}, \tag{87}$$

where $\psi^\dagger$ is the value of the moment after the binary collision, ψ is the value before the collision, and $\mathbf{w}$ is the difference in the vertical velocities of the two particles. The vector $\mathbf{k}$ is the unit vector directed along the line joining the centers of the two particles directed from the particle with velocity u_y to the particle with velocity u_{1y}, and the integral is carried out for particles with $\mathbf{w}\cdot\mathbf{k}\geq 0$ which are traveling towards each other. The third moments of the velocity distribution can easily be evaluated by integrating Eq. (87),

$$\langle n u_y^3\rangle = \int_\infty^y dy \frac{\partial_c n u_y^2}{\partial t}, \tag{88}$$

$$\langle n u_y u_x^2\rangle = \int_\infty^y dy \frac{\partial_c n u_x^2}{\partial t}. \tag{89}$$

These can be carried out analytically, and the final expressions for the third moments of the velocity distribution are

$$\langle u_y^3\rangle = -\frac{32nrT^{3/2}}{15\sqrt{\pi}}\exp(-2gy/T), \tag{90}$$

$$\langle u_y u_x^2\rangle = \frac{32nrT^{3/2}}{15\sqrt{\pi}}\exp(-2gy/T). \tag{91}$$

The above calculation shows that the magnitudes of the third moments are $(nr)T^{3/2}$, which is $O(\delta_I\epsilon_I T^{3/2})$. The reason for the signs of the above moments can be explained as follows. The third moment $\langle u_y^3\rangle$ at a height y^* is only due to particles that have encountered a binary collision above a height y^*, since particles which have collided below the height y^* have velocities symmetric about $u_y^*=0$ on their upward and downward paths. For particles that have undergone a collision above the height y^*, the magnitude of the velocity in the y direction on the downward trajectory is, on average, lower than its value in the absence of a collision, since the binary collision transfers energy, on average, from the vertical to the horizontal direction. Consequently, the third moment $\langle u_y^3\rangle$ is negative. Energy conservation during a binary collision requires that the moment $\langle u_y u_x^2\rangle$ is equal in magnitude and opposite in direction to $\langle u_y^3\rangle$.

IV. CONCLUSIONS

The velocity distribution function for a single particle colliding with a vibrating surface was calculated in Sec. II in the limit where the mean square velocity of the particle is large compared to the amplitude of the velocity of the vibrating surface, and the period of oscillation of the vibrating surface is small compared to the time between successive collisions so that there is no correlation in the velocity of the vibrating surface during successive collisions. In this limit, the change in energy of the particle during a collision is small compared to the energy of the particle, and an ordinary differential equation was derived for the distribution function. Two dissipation mechanisms—inelastic collisions and fluid drag—and two type of amplitude functions for the vibrating surface—symmetric and asymmetric—were considered. The important results are as follows.

(i) For a system where the dissipation is due to inelastic collisions and the amplitude function is symmetric, the distribution function is a one-dimensional Maxwell-Boltzmann distribution and the mean square velocity scales as $[2\langle U^2\rangle_S/(1-e)]$. This is in agreement with the present simulation results, as well as those of Warr *et al.* [14].

(ii) When the dissipation is due to inelastic collisions and the amplitude function is asymmetric, the distribution function is bimodal with sharp peaks at $[\pm 2\langle U\rangle_S/(1-e)]$ at the vibrating surface, and the width of each of these peaks is $O[\langle U\rangle_S/(1-e)^{1/2}]$.

(iii) When the dissipation of energy is due to a drag force that is a linear function of the particle velocity and the amplitude function is symmetric, the distribution function is very different from a Gaussian distribution. It has a maximum at the origin, and decays proportional to $\int_{u_y}^{\infty} du_y \exp(-u_y^3/9)$ in the limit $u_y\to\infty$. The mean square velocity scales as $(\langle U^2\rangle_S g/\mu)^{2/3}$ in this case.

(iv) When the dissipation is due to a drag force that is a linear function of the particle velocity, and the amplitude function is asymmetric, the distribution function at the vibrating surface is sharply peaked about $\pm(3\langle U\rangle_S g/\mu)^{1/2}$. The width of the peaks scales as $\langle U\rangle_S$.

(v) When the dissipation is due to a drag law of the form $(du_y/dt)=-\mu_m u_y|u_y|^{m-1}$ and the amplitude function is symmetric, the mean square velocity scales as $(\langle U^2\rangle_S g/\mu_m)^{2/(m+2)}$ and the distribution function is very different from a Gaussian distribution.

(vi) When the dissipation is due to a drag law of the form $(du_y/dt)=-\mu_m u_y|u_y|^{m-1}$ and the amplitude function is asymmetric, the distribution function is sharply peaked about $\pm[(n+2)\langle U\rangle_S g/\mu_m]^{1/(m+1)}$ and the width of the peaks scales as $\langle U\rangle_S$.

In their experiments, Warr *et al.* [7] reported that the mean square velocity scaled as $\langle U^2\rangle^{0.52}$, and speculated that the discrepancy between their experiments and theory could be due to fluid drag or due to the small sample size. The present analysis indicates that fluid drag reduces the scaling exponent for the mean square velocity by $\frac{2}{3}$ if the drag law is linear, and $\frac{1}{2}$ if the drag law is quadratic for turbulent flow. Though this is close to the exponent observed by Warr *et al.*, the semiquantitative analysis carried out by the authors indicates that turbulent drag is not the mechanism causing the change in exponent.

The velocity distribution that includes the first effect of binary collisions for a system where the dissipation is due to inelastic collisions was determined in Sec. III. An asymptotic expansion was used in the small parameter $\delta_I=2nr/(1-e)$, where n is the number of particles per unit width of the bed and r is the particle radius. Certain assumptions were made regarding the rate of accumulation and depletion of particles in velocity space in order to simplify the calculations, and it was shown that the error made due to these assumptions is $O(\delta_I)$ smaller than the leading order fluxes. With these assumptions, an analytical form of the distribution function was derived in the region $u_x\ll\sqrt{T}$. In this region, it was found that the distribution function has a power-law divergence proportional to $|u_x|^{(c\delta_I-1)}$, where the constant $c=(\sqrt{2}/\pi)$. Therefore, the distribution function is integrable in the limit $u_x\rightarrow 0$, and this distribution function converges to the single particle distribution function in the limit $\delta_I\rightarrow 0$. The distribution function in the region $u_x\sim\sqrt{T}$ was determined using an expansion in appropriate orthogonal polynomials in the u_x and u_y coordinates. The nontrivial second and third moments of the velocity distribution were evaluated by averaging the Boltzmann equation, and it was found that the mean square of the horizontal velocity is $O(\delta_I T)$, and the third moments scale as $O(\delta_I\epsilon_I T^{3/2})$, where T is the mean square velocity in the vertical direction.

Though the present analysis is restricted to the limit of small perturbations, it provides a first step towards understanding the effect of the amplitude function and dissipation mechanism on the velocity distribution function. There have been earlier studies [15,16] which have reported the effect of scaling on the amplitude function, but the present analysis indicates that the form of the velocity distribution function is also dependent on the form of the amplitude function of the vibrating surface. In addition, the form of the distribution function and the scaling of the temperature are sensitive to the type of energy dissipation as well. The analysis also provides useful information for more approximate theories which could be used over a range of densities, such as the requirement that the anisotropy in the velocity distribution function should be $O(\delta_I)$ as the single particle limit is approached, and the third moment of the vertical velocity should be $O(\delta_I\epsilon_I T^{3/2})$ in this limit.

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